

GEOMETRY OF THE BANACH SPACES $C(\beta\mathbb{N} \times K, X)$ FOR COMPACT METRIC SPACES K

DALE E. ALSPACH AND ELÓI MEDINA GALEGO

ABSTRACT. A classical result of Cembranos and Freniche states that the $C(K, X)$ spaces contains a complemented copy of c_0 whenever K is an infinite compact Hausdorff space and X is an infinite dimensional Banach space. This paper takes this result as a starting point and begins a study of the conditions under which the spaces $C(\alpha)$, $\alpha < \omega_1$ are quotients of or complemented in spaces $C(K, X)$.

In contrast to the c_0 result, we prove that if $C(\beta\mathbb{N} \times [1, \omega], X)$ contains a complemented copy of $C(\omega^\omega)$ then X contains a copy of c_0 . Moreover, we show that $C(\omega^\omega)$ is not even a quotient of $C(\beta\mathbb{N} \times [1, \omega], l_p)$, $1 < p < \infty$.

We then completely determine the separable $C(K)$ spaces which are isomorphic to a complemented subspace or a quotient of the $C(\beta\mathbb{N} \times [1, \alpha], l_p)$ spaces for countable ordinals α and $1 \leq p < \infty$. As a consequence, we obtain the isomorphic classification of the $C(\beta\mathbb{N} \times K, l_p)$ spaces for infinite compact metric spaces K and $1 \leq p < \infty$. Indeed, we establish the following more general cancellation law. Suppose that the Banach space X contains no copy of c_0 and K_1 and K_2 are infinite compact metric spaces, then the following statements are equivalent:

- (1) $C(\beta\mathbb{N} \times K_1, X)$ is isomorphic to $C(\beta\mathbb{N} \times K_2, X)$
- (2) $C(K_1)$ is isomorphic to $C(K_2)$.

These results are applied to the isomorphic classification of some spaces of compact operators.

1. INTRODUCTION

The isomorphic classification of the separable spaces of continuous functions on a compact Hausdorff space was completed in 1966 when Milutin, [22], [24], showed that there was a single isomorphism class for the continuous functions on uncountable compact metric spaces. For general compact Hausdorff spaces some work has been done in

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special cases, e.g., [15] or [18], but, unlike the isometric case which is completely determined by the Banach-Stone theorem, extended in [2] and [5], the isomorphic classification seems hopeless.

In this paper we consider a special class of compact Hausdorff spaces but allow the range space to be a Banach space instead of \mathbb{R} . Thus we study the spaces $C(K, X)$ of continuous functions from K into X where X is a Banach space, K is a compact Hausdorff space and the norm of an element f is $\|f\| = \sup_{k \in K} \|f(k)\|_X$. Usually K will be a compact metric space or a product of a compact metric space and the Stone-Cech compactification of the natural numbers, $\beta\mathbb{N}$. Our interest in $\beta\mathbb{N}$ stems from application of some of the results to the following question, [11, Problem 4.2.2]. From now on $\mathcal{K}(X, Y)$ denotes the space of compact operators from X to another Banach space Y and $[1, \alpha]$ is the compact Hausdorff space of ordinals between 1 and α in the order topology.

Problem 1.1. *Classify, up to an isomorphism, the spaces of compact operators $\mathcal{K}(l_1, C([1, \alpha], l_p))$, where $\alpha \geq \omega$ and $1 \leq p < \infty$.*

This problem covers some cases remaining from the development in [11], [12], [13], [14] and [25], of the isomorphic classification of some spaces of compact operators. We give the solution to above problem in the case where α is countable. The connection to the spaces $C(K, X)$ comes through the injective tensor product. Notice that that since l_1 has the approximation property, by [7, Proposition 5.3] we know that for every ordinal α and $1 \leq p < \infty$,

$$\mathcal{K}(l_1, C([1, \alpha], l_p)) \sim C(\beta\mathbb{N} \times [1, \alpha], l_p).$$

The notation for the spaces is a bit cumbersome so we will shorten some expressions. When the context clearly requires a compact Hausdorff space we will write α rather than $[1, \alpha]$. In particular, $C(\alpha, X) = C([1, \alpha], X)$. If $X = \mathbb{R}$, we will write $C(K)$ rather than $C(K, \mathbb{R})$. We will also adopt some standard notational conventions from Banach space theory. We write $X \sim Y$ when the Banach spaces X and Y are isomorphic, $Y \hookrightarrow X$ when X contains a copy of Y , that is, a subspace isomorphic to Y , $Y \overset{c}{\hookrightarrow} X$ if X contains a complemented copy of Y and $X \twoheadrightarrow Y$ when Y is a quotient of X . For other notation and terminology we refer the reader to [16] and [20].

In $C(K, X)$ an obvious part of the difficulty with the isomorphic classification is that structures in X can be used to find an alternate compact Hausdorff space K_1 so that $C(K_1, X)$ is isomorphic to $C(K, X)$. A second difficulty is that structures may arise that are present in neither

$C(K)$ nor X . Consider the following result, which was obtained independently by Cembranos [6, Main Theorem] and Freniche [9, Corollary 2.5].

Theorem 1.2. *Let K be an infinite compact Hausdorff space and X an infinite dimensional Banach space. Then*

$$c_0 \xhookrightarrow{c} C(K, X).$$

Consequently, both $C(\beta\mathbb{N}, C(\beta\mathbb{N}))$ and $C(\beta\mathbb{N}, \ell_2)$ contain a complemented subspace isomorphic to c_0 despite the fact that neither $C(\beta\mathbb{N})$ nor ℓ_2 have complemented copies of c_0 .

It is natural to ask if there are other $C(K)$ spaces for which the analogous result holds. The next more complicated $C(K)$ space is $C(\omega^\omega)$. Our first result gives a negative answer in this case. We prove that even when $C(K)$ contains a complemented copy of c_0 and X is an infinite dimensional Banach space, $C(K, X)$ may not contain a complemented copy of $C(\omega^\omega)$. Indeed, it is easy to see that $C(\beta\mathbb{N} \times \omega)$ contains a complemented copy of c_0 . However, in Section 3, we prove the following.

Theorem 3.2. Let X be a Banach space. Then

$$C(\omega^\omega) \xhookrightarrow{c} C(\beta\mathbb{N} \times \omega, X) \implies c_0 \hookrightarrow X.$$

We then extend Theorem 3.2 to larger ordinals by using that result and the structure of the ordinals.

Theorem 3.5. Let X be a Banach space containing no copy of c_0 , K an infinite compact metric space and $0 \leq \alpha < \omega_1$. Then

$$C(K) \xhookrightarrow{c} C(\beta\mathbb{N} \times \omega^{\omega^\alpha}, X) \iff C(K) \sim C(\omega^{\omega^\xi}) \text{ for some } 0 \leq \xi \leq \alpha.$$

In Section 4 we turn our attention to spaces of the form $C(\beta\mathbb{N} \times \alpha, X)$ where X satisfies some geometrical properties, (\dagger) and (\ddagger) , that are modeled on simple properties of ℓ_p , $1 \leq p < \infty$. In particular we get that $C(\omega^\omega)$ is not a quotient of $C(\beta\mathbb{N} \times \omega, \ell_p)$, $1 < p < \infty$.

Theorem 4.3. Suppose that X is a Banach space satisfying the daggers. Then $C(\omega^\omega)$ is not a quotient of $C(\omega \times \beta\mathbb{N}, X)$.

Similar to the way we obtained Theorem 3.5 from Theorem 3.2 in Section 4, we also extend Theorem 4.3 by proving.

Theorem 4.6. Let K be an infinite compact metric space and $0 \leq \alpha < \omega_1$. Then

$$C(\beta\mathbb{N} \times \omega^{\omega^\alpha}, \ell_p) \twoheadrightarrow C(K) \iff C(K) \sim C(\omega^{\omega^\xi}) \text{ for some } 0 \leq \xi \leq \alpha.$$

The next section concerns the isomorphic classification of the spaces $C(\beta\mathbb{N} \times \alpha, l_p)$. Our results also provide us with immediate information about the isomorphic classifications of a wider class of Banach spaces. Namely, the $C(\beta\mathbb{N} \times K, X)$ spaces, where X contains no copy of c_0 and K is a metrizable compact space, that is, $C(K)$ is a separable space. Indeed, in Section 3 we prove the following cancellation law which is the main application of the results of the paper. The case $X = l_p$, $1 \leq p < \infty$, gives the solution to Problem 1.1.

Theorem 3.7. Let X be a Banach space containing no copy of c_0 . Then for any infinite compact metric spaces K_1 and K_2 we have

$$C(\beta\mathbb{N} \times K_1, X) \sim C(\beta\mathbb{N} \times K_2, X) \iff C(K_1) \sim C(K_2).$$

Moreover, in Section 5, we accomplish the isomorphic classification of the spaces $C(\beta\mathbb{N} \times K, l_p)$ by considering also the case where K is finite. In order to do this, we first prove a general result about the spaces of compact operators $\mathcal{K}(\ell_p(X), \ell_q(Y))$ (Theorem 5.3). From that we deduce the following.

Theorem 5.4. Let $1 \leq p < \infty$. Then

$$C(\beta\mathbb{N} \times \omega, l_p) \sim C(\beta\mathbb{N}, l_p).$$

Finally, in Section 6, we pose some elementary questions which this work raises.

2. PRELIMINARIES

In this section we recall some results that we will use in the sequel.

In 1920 Mazurkiewicz and Sierpiński showed that if K is a countable compact metric space then it is homeomorphic to an interval of ordinals $[1, \alpha]$ with $\omega \leq \alpha < \omega_1$ [21]. This was used in the isomorphic classification of the $C(\alpha)$ spaces, $\omega \leq \alpha < \omega_1$, obtained in 1960 by Bessaga and Pełczyński. They showed that if $\omega \leq \alpha \leq \beta < \omega_1$ then $C(\alpha)$ is isomorphic to $C(\beta)$ if and only if $\beta < \alpha^\omega$, see [4] and [24]. In particular this means that the spaces $C(\omega^{\omega^\gamma})$, for $0 \leq \gamma < \omega_1$, are a complete set of representatives of the isomorphism classes of $C(K)$ where K is a countably infinite, compact metric space.

Bessaga and Pełczyński actually prove some things for the case of $C(K, X)$, where X is a Banach space.

Proposition 2.1. *Suppose X is a Banach space and α is an infinite ordinal. Then $C(\alpha, X)$ is isomorphic to*

$$(1) \ C_0(\alpha, X) = \{f \in C(\alpha, X) : f(\alpha) = 0\}$$

and to

$$(2) \ C(\omega^{\omega^\beta}, X) \text{ whenever } \omega^{\omega^\beta} \leq \alpha < \omega^{\omega^{\beta+1}}.$$

Thus for some Banach spaces X there may be fewer isomorphism classes for the spaces $C(K, X)$ with K countable, compact metric, than for the case $X = \mathbb{R}$. That is what happens for Banach spaces which are isomorphic to their squares or to the c_0 -sum of infinitely many copies of the space. Indeed, for ℓ_p , $1 \leq p < \infty$, the finite ordinals all yield the same space; for c_0 all of the spaces $C(\alpha, c_0)$, $\alpha < \omega^\omega$, are isomorphic.

Remark 2.2. The order structure on the spaces of ordinals make it easy to find contractively complemented subspaces of $C(\omega^\alpha)$ isometric to $C(\omega^\beta)$ for $\beta < \alpha$. Indeed if A is a closed subset of $[1, \omega^\alpha]$ and $A^{(1)}$ is the set of non-isolated points of A , we can define a subspace Y of $C(\omega^\alpha)$ isometric to $C(A)$ by

$$Y = \{f \in C(\omega^\alpha) : f(\gamma) = f(\xi), \forall \gamma \text{ such that } \sup\{\rho < \xi : \rho \in A\} < \gamma < \xi \text{ and } \xi \in A \setminus A^{(1)}\}.$$

We can define a projection onto Y by restricting to A and then extending by the formula in the definition of Y , i.e.,

$$Lg(\gamma) = g(\xi) \text{ for all } \gamma \text{ such that } \sup\{\rho < \xi : \rho \in A\} < \gamma < \xi \text{ and } \xi \in A \setminus A^{(1)}.$$

For $\gamma > \sup A$ let $Lg(\gamma) = 0$.

The spaces c_0 and $C(\beta\mathbb{N})$ play a prominent role in this paper so we now recall some important properties of these spaces. Bessaga and Pełczyński made a study of c_0 in [3] and introduced the notion of a weakly unconditionally Cauchy sequence (w.u.c.). A sequence (x_n) in a Banach space X is said to be a w.u.c. if and only if for every $x^* \in X^*$, $\sum_n |x^*(x_n)| < \infty$. A sequence equivalent to standard basis of c_0 is clearly a w.u.c. We will use the following result from their paper.

Proposition 2.3. *Suppose that X is a Banach space which has no subspace isomorphic to c_0 then every w.u.c. in X is unconditionally converging. Consequently, if (x_n) is a w.u.c. in X , $\lim_n \|x_n\| = 0$.*

$C(\beta\mathbb{N})$ is isometric to $\ell_\infty = \ell_\infty(\mathbb{N})$, the space of bounded sequences with the supremum norm. For any non-empty index set Γ , $\ell_\infty(\Gamma)$ is injective, i.e., it is complemented in any space which contains it. c_0 is separably injective, i.e., it is complemented in any separable Banach space that contains it. c_0 is not complemented in ℓ_∞ and in fact the only infinite dimensional complemented subspaces of c_0 or ℓ_∞ are isomorphs of the whole space, [20, pages 54 and 57]. ℓ_∞ is an example of a Grothendieck space, i.e., any weak* convergent sequence in the dual is actually weakly convergent [8, page 179]. Actually this is the

essential property of $C(\beta\mathbb{N})$ that we use. Except in one or two cases, e.g., Theorem 5.5, the results could be rewritten with $C(K)$ which is Grothendieck space in place of $C(\beta\mathbb{N})$.

While $C(\beta\mathbb{N})$ and ℓ_∞ are isometric, for many infinite dimensional Banach space X , $C(\beta\mathbb{N}, X)$ is not isomorphic to $\ell_\infty(X) = \{(x_n) : x_n \in X \text{ for all } n, \|(x_n)\| = \sup_n \|x_n\|_X < \infty\}$. This is the case if X does not contain a complemented subspace isomorphic to c_0 since, by [19], $\ell_\infty(X)$ only contains a complemented subspace isomorphic to c_0 if X does.

We identify $C(\beta\mathbb{N}, X)^*$ with the space of X^* -valued regular Borel measures on $C(\beta\mathbb{N})$ with $\|\mu\| = \sup \sum_n \|\mu(A_n)\|_{X^*}$ where the supremum is over all partitions of $\beta\mathbb{N}$ into disjoint clopen sets $\{A_n\}$, [8, page 182]. Moreover if (μ_n) is a weak* converging sequence of measures with limit μ , then for any clopen set A , $(\mu_n(A))$ converges weak* in X^* to $\mu(A)$.

It will be convenient at times to shift point of view as to the underlying compact Hausdorff space and the range space. Thus we will use the fact that $C(K_1 \times K_2, X)$ is isomorphic to $C(K_1, C(K_2, X))$ and to $C(K_2, C(K_1, X))$ where K_1 and K_2 are compact Hausdorff spaces. Also because $C(K, X)$ is isometric to the injective tensor product $C(K) \overset{\vee}{\otimes} X$, we may replace K by K_1 if $C(K)$ is isomorphic to $C(K_1)$.

Let $\max_\sigma\{\beta_1, \beta_2\}$ be largest ordinal $\beta = \gamma_1 + \alpha_1 + \gamma_2 + \alpha_2 + \dots + \gamma_k + \alpha_k$ obtained by writing $\beta_1 = \gamma_1 + \gamma_2 + \dots + \gamma_k$ and $\beta_2 = \alpha_1 + \alpha_2 + \dots + \alpha_k$, where $\gamma_j \geq 0$ and $\alpha_j \geq 0$ for all j . This can also be obtained by writing the ordinals β_1 and β_2 in terms of prime components and arranging the terms of the sum in decreasing order.

The topological results in [21] are based on the notion of derived set. Recall that $K^{(0)} = K$. For any ordinal α , $K^{(\alpha+1)}$ is the set of non-isolated points in $K^{(\alpha)}$, and for a limit ordinal β , $K^{(\beta)} = \bigcap_{\alpha < \beta} K^{(\alpha)}$. We will only use this with countable compact spaces and will refer to the smallest ordinal α such that $K^{(\alpha)} \neq \emptyset$ and $K^{(\alpha+1)} = \emptyset$ as the derived order of K .

Lemma 2.4. *Let K_1 and K_2 be countable compact metric spaces and β_1, β_2 be countable ordinals such that $K_1^{(\beta_1)}$ and $K_2^{(\beta_2)}$ are finite non-empty sets. Then $(K_1 \times K_2)^{(\max_\sigma\{\beta_1, \beta_2\})}$ is a finite non-empty set.*

Sketch of Proof. First we can assume that $K_1^{(\beta_1)}$ and $K_2^{(\beta_2)}$ are singletons, k_1 and k_2 , respectively. The proof is an induction on β_2 and for each β_2 on β_1 , $0 \leq \beta_1 \leq \beta_2$. The result is clear for $\beta_2 = 0, 1$ and $\beta_1 = 0, 1$ and for all $\beta_2 < \omega_1$ and $\beta_1 = 0$. Assume $1 < \beta_2$, $0 < \beta_1 \leq \beta_2$

and that the result holds for K_2 of order $\beta < \beta_2$ and K_1 of order $\gamma \leq \beta$ and for $\beta = \beta_2$ and $\gamma < \beta_1$.

To see that

$$(K_1 \times K_2)^{(\max_{\sigma}\{\beta_1, \beta_2\})} = \{(k_1, k_2)\},$$

notice that by the induction assumption, if $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$, $m_1 \in K_1^{(\alpha_1)} \setminus K_1^{(\alpha_1+1)}$ and $m_2 \in K_2^{(\alpha_2)} \setminus K_2^{(\alpha_2+1)}$, then

$$(m_1, m_2) \in (C_1 \times C_2)^{(\max_{\sigma}\{\alpha_1, \alpha_2\})},$$

where C_1 and C_2 , are appropriately chosen clopen subsets of K_1 and K_2 , respectively. For $i = 1, 2$ write $K_i \setminus \{k_i\}$, as a disjoint union of clopen sets $C_{i,n}$, derived order $\beta_i - 1$ for all n or derived order $\beta_{i,n}$ and $(\beta_{i,n})$ increasing to β_i . Each set $C_{1,n} \times [1, \beta_2]$ and $[1, \beta_1] \times C_{2,n}$ satisfies the induction hypothesis (possibly symmetrized). Notice that

$$\cup_n C_{1,n} \times [1, \beta_2] \cup [1, \beta_1] \times C_{2,n} = K_1 \times K_2 \setminus \{(k_1, k_2)\}.$$

There are four cases to check. It is easy to see that

$$\max_{\sigma}\{\beta_1, \beta_2 - 1\} + 1 = \max_{\sigma}\{\beta_1, \beta_2\},$$

in the first case, and $\max_{\sigma}\{\beta_1, \beta_{2,n}\}$ increases to $\max_{\sigma}\{\beta_1, \beta_2\}$ in the second case. The other two case are similar. \square

The lemma shows that we do not really gain anything from simple manipulations of compact metric spaces. Indeed, for countable ordinals α and γ , we have that

$$C(\beta\mathbb{N} \times \omega^{\omega^{\alpha}}, C(\omega^{\omega^{\gamma}}, X)) \sim C(\beta\mathbb{N} \times \omega^{\omega^{\alpha}} \times \omega^{\omega^{\gamma}}, X).$$

However

$$\omega^{\omega^{\max\{\alpha, \gamma\}}} \leq \omega^{\max_{\sigma}\{\omega^{\alpha}, \omega^{\gamma}\}} \leq \omega^{\omega^{\max\{\alpha, \gamma\}} 2}$$

and thus

$$C(\beta\mathbb{N} \times \omega^{\omega^{\alpha}} \times \omega^{\omega^{\gamma}}, X) \sim C(\beta\mathbb{N} \times \omega^{\omega^{\max\{\alpha, \gamma\}}}, X).$$

In a series of papers from the 1970's the first author developed some tools for working with subspaces of $C(K)$ spaces isomorphic to $C(\alpha)$. Some of the proofs in this paper are motivated in part by that work and versions of some of the technical tools will be needed here. The first is similar to [1, Lemma 2.5].

Lemma 2.5. *Given a positive integer k and $\epsilon > 0$ there is a positive integer n such that if $(x_{\alpha}^*)_{\alpha \leq \omega^n}$ is a sequence in the unit ball of the dual of a Banach space X such that the function $\alpha \rightarrow x_{\alpha}^*$ is a order to weak* continuous map, then there is a closed subset B of $[1, \omega^n]$, order isomorphic and homeomorphic via the order isomorphism to $[1, \omega^k]$ such that for all $\beta, \beta' \in B$, $|||x_{\beta}^*|| - ||x_{\beta'}^*||| < \epsilon$.*

The next lemma is a vector valued version of a typical construction of a sequence of disjointly supported functions normed by a sequence of measures.

Lemma 2.6. *Suppose that X is a Banach space, C, D are positive constants, K is a compact Hausdorff space, (μ_n) is a sequence of elements of $C(K, X)^*$ represented as X^* valued measures on K , with $\|\mu_n\| \leq C$ for all n , and (g_n) is a sequence of norm at most D elements of $C(K, X)$ such that*

$$\int g_n d\mu_n \geq 1,$$

for all n and $\|g_n(t)\| \rightarrow 0$ as $n \rightarrow \infty$ for all $t \in K$. Then for any $\epsilon > 0$ there are an infinite subset M of \mathbb{N} and open subsets $(G_m)_{m \in M}$ of K such that $\overline{G_m} \cap \overline{G_j} = \emptyset$ if $m \neq j$, and

$$\int g_m 1_{G_m} d\mu_m > 1 - \epsilon,$$

for all $m \in M$.

Proof. Let $\epsilon > 0$, and $\epsilon_k = \epsilon/2^{k+2}$, for $k \in \mathbb{N}$. Choose $\rho > 0$ such that $\rho < \epsilon/(4C)$. Then

$$\int g_n 1_{\{t: \|g_n(t)\| \geq \rho\}} d\mu_n > 1 - \frac{\epsilon}{4}, \quad \forall n, \quad 1 \leq n < \omega.$$

Because $\|g_n(t)\|$ converges to 0, $\int \|g_n(t)\| d|\mu_i|(t)$ converges to 0 for each i .

Thus for $i = 1, 2, \dots, j_1$, $j_1 > C/\epsilon_1$, there are infinite subsets $\mathbb{N} \supset N_1 \supset \dots \supset N_{j_1}$, $n_1 = 1$, and $n_{i+1} = \min N_i$ such that

$$\sum_{n \in N_i} |\mu_{n_i}|(\{t: \|g_n(t)\| > \rho/2\}) \leq \sum_{n \in N_i} (2/\rho) \int \|g_n(t)\| d|\mu_{n_i}|(t) < \epsilon/(4C).$$

For $i = 1, 2, \dots, j_1$, let

$$A_i = \{t: \|g_{n_i}(t)\| \geq \rho\} \setminus \cup_{n \in N_i} \{t: \|g_n(t)\| > \rho\}.$$

Find disjoint open sets H_1, H_2, \dots, H_{j_1} such that

$$\{t: \|g_i(t)\| > \rho/2\} \supset H_i \supset A_i,$$

for each i . Because $\|\mu_m\| \leq C$ for all m and the sets (H_i) are disjoint, for some infinite subset M_1 of N_{j_1} and some i_1 , $1 \leq i_1 \leq j_1$,

$$|\mu_m|(H_{i_1}) < \epsilon/8 = \epsilon_1,$$

for all $m \in M_1$. Let $m_1 = n_{i_1}$ and G_{m_1} be an open set containing A_{i_1} such that $\overline{G_{m_1}} \subset H_{i_1}$. m_1 is the first element of M and μ_{m_1} and G_{m_1} are the corresponding measure and open set pair.

Let $K_1 = K \setminus H_{i_1}$. Now notice that if we consider $(\mu_m|_{K_1})_{m \in M_1}$ and $(g_m|_{K_1})_{m \in M_1}$ we have the original situation with 1 replaced by $1 - \epsilon/8$ as the lower bound on

$$\int_{K_1} g_m d\mu_m.$$

Thus repeating the argument above with ϵ_2 but choosing open sets as open subsets of $K \setminus \overline{G_{m_1}}$ rather than K_1 (and hence open in K), we get μ_{m_2} and G_{m_2} with $G_{m_1} \cap G_{m_2} = \emptyset$.

Continuing in this way we can construct the required indices and open sets. □

3. COMPLEMENTED SEPARABLE $C(K)$ SUBSPACES OF $C(\beta\mathbb{N} \times \alpha, X)$

It is clear that for any Banach space X , $C(\beta\mathbb{N} \times \omega, X)$ contains a complemented copy of c_0 . This section is devoted to proving that c_0 is, up to an isomorphism, the only separable $C(K)$ space which is complemented in $C(\beta\mathbb{N} \times \omega, X)$ whenever X contains no copy of c_0 . This is a direct consequence of Theorem 3.2.

The next lemma is a technical analog of a result of Bessaga and Pełczyński, see [3, Theorem 4].

Lemma 3.1. *Suppose that X and Y are Banach spaces and that T is an operator from $C(\beta\mathbb{N}, X)$ into Y . If there exist an element f of $C(\beta\mathbb{N}, X)$, $\delta > 0$, a sequence of disjoint non-empty clopen subsets (G_n) of $\beta\mathbb{N}$ and a sequence of X^* -valued measures (μ_n) contained in $T^*(B_{Y^*})$ such that*

$$|\int f 1_{G_n} d\mu_n| > \delta,$$

for all n , then there is a subspace Z of $C(\beta\mathbb{N}, X)$ such that Z is isomorphic to ℓ_∞ and $T|_Z$ is an isomorphism into Y .

Proof. Let $|\mu_n|$ denote the real-valued total variation measure induced by μ_n . Observe that the sequence of pairs $(|\mu_n|, G_n)$ satisfy the hypotheses of Rosenthal's disjointness [24, Lemma 1, page 18]. Therefore there exists a subsequence $(|\mu_n|, G_n)_{n \in M}$ such that for all $n \in M$,

$$\sum_{j \in M, j \neq n} |\mu_n|(G_j) < \delta/(8\|f\|).$$

We also need to have

$$|\mu_n|(\overline{\cup_{j \in M} G_j} \setminus \cup_{j \in M} G_j) < \delta/(8\|f\|),$$

for all $n \in M$. If for some $n \in M$,

$$|\mu_n|(\overline{\cup_{j \in M} G_j} \setminus \cup_{j \in M} G_j) \geq \delta/(8\|f\|),$$

we can argue as follows. Partition M into an infinite number of infinite sets (M_k) . If for some k , for all $n \in M_k$,

$$|\mu_n|(\overline{\cup_{j \in M_k} G_j} \setminus \cup_{j \in M_k} G_j) < \delta/(8\|f\|),$$

we can continue with M_k in place of M . If not, for each k choose $n_k \in M_k$ such that

$$|\mu_{n_k}|(\overline{\cup_{j \in M_k} G_j} \setminus \cup_{j \in M_k} G_j) \geq \delta/(8\|f\|).$$

Let $M^1 = \{n_k : k \in \mathbb{N}\}$. Observe that for all k ,

$$(\overline{\cup_{j \in M_k} G_j} \setminus \cup_{j \in M_k} G_j) \cap \overline{\cup_{j \in M^1} G_j} = \emptyset.$$

Now if for all $n \in M^1$,

$$|\mu_n|(\overline{\cup_{j \in M^1} G_j} \setminus \cup_{j \in M^1} G_j) < \delta/(8\|f\|),$$

we can use M^1 in place of M . If not, notice that for all $n \in M^1$,

$$\|\mu_n|_{\overline{\cup_{j \in M^1} G_j}}\| \leq \|\mu_n\| - \delta/(8\|f\|).$$

We can split M^1 into infinitely many infinite sets and repeat the previous argument. Each time this process reduces the norm of the part of μ_n under consideration by $\delta/(8\|f\|)$. Thus in at most $\|f\| \|T\| 8/\delta$ repetitions of the argument we will find the required infinite set M such that for all $n \in M$,

$$|\mu_n|(\overline{\cup_{j \in M} G_j} \setminus \cup_{j \in M} G_j) < \delta/8,$$

and

$$\sum_{j \in M, j \neq n} |\mu_n|(G_j) < \delta/(8\|f\|).$$

Let Z be given by

$$\{g \in C(\beta\mathbb{N}, X) : g(t) = 0 \ \forall t \notin \overline{\cup_{n \in M} G_n}, g1_{G_n} = c_n f 1_{G_n}, c_n \in \mathbb{R} \ \forall n \in M\}.$$

Because the range of f is compact, for any bounded sequence of real numbers $(c_n)_{n \in M}$, the function h defined on \mathbb{N} by

$$h(k) = \begin{cases} 0 & \text{if } k \notin \cup_{m \in M} G_m, \\ c_n f(k) & \text{if } k \in G_n \text{ and } n \in M. \end{cases}$$

is in $\ell_\infty(X)$ with relatively compact range and hence extends continuously to some function H on $\beta\mathbb{N}$ with values in the symmetric radial hull of $\|(c_n)\|_\infty$ times the range of f . Moreover because \mathbb{N} is dense, the extension is unique and must agree with $c_n f 1_{G_n}$ on G_n for all $n \in M$, and be 0 on the closure of

$$\{k : k \in \mathbb{N}, k \notin \cup_{n \in M} G_n\}.$$

Therefore Z is isomorphic to ℓ_∞ , and for (c_n) and H as above,

$$(\delta/\|T\|)\|(c_n)_{n \in M}\|_\infty \leq \inf_{n \in M} \|f1_{G_n}\| \|(c_n)_{n \in M}\|_\infty \leq \|H\| \leq \|f\| \|(c_n)_{n \in M}\|_\infty.$$

Continuing with the same notation, we can get a lower bound on $\|TH\|$ as follows. Observe that for each $n \in M$,

$$|\int H d\mu_n|$$

is greater than or equal to

$$|\int H1_{G_n} d\mu_n| - \sum_{j \in M, j \neq n} |\mu_n|(G_j) \|f\| |c_j| - |\mu_n|(\overline{\cup_{j \in M} G_j} \setminus \cup_{j \in M} G_j) \|f\| \|(c_j)\|_\infty,$$

which in turn is greater than or equal to

$$|c_n| |\int f1_{G_n} d\mu_n| - \|(c_j)_{j \in M}\|_\infty \delta/4 \geq \delta(|c_n| - \|(c_j)_{j \in M}\|_\infty/4).$$

Taking the supremum over n and noting that $\mu_n \in T(B_{Y^*})$ completes the proof. \square

Theorem 3.2. *Let X be a Banach space. Then*

$$C(\omega^\omega) \xhookrightarrow{c} C(\beta\mathbb{N} \times \omega, X) \implies c_0 \hookrightarrow X.$$

Proof. Assume that X does not contain a subspace isomorphic to c_0 . We will show that the existence of the complemented subspace isomorphic to $C(\omega^\omega)$ produces the situation in the hypothesis of the previous lemma. First we will reduce to a simplified situation. By Proposition 2.1.1, $C(\beta\mathbb{N} \times \omega, X)$ is isomorphic to $C_0(\omega \times \beta\mathbb{N}, X)$, i.e., the c_0 -sum of $C(\beta\mathbb{N}, X)$. Assume now that T is a projection from $C_0(\omega \times \beta\mathbb{N}, X)$ onto a subspace Y isomorphic to $C(\omega^\omega)$. Let $S : Y \rightarrow C(\omega^\omega)$ be the isomorphism and suppose, without loss of generality, that $\|S\| \leq 1$. Then T^*S^* is an isomorphism with lower bound some $\epsilon > 0$, i.e.,

$$\|T^*S^*z\| \geq \epsilon\|z\|,$$

for all $z \in C(\omega^\omega)^*$. Choose N by Lemma 2.5 so that for $n > 8\|T\|$, there exists a subfamily $\{\mu_\beta : \beta \leq \omega^n\}$ of $\{T^*S^*\delta_\gamma : \gamma \leq \omega^N\}$ such that $\beta \rightarrow \mu_\beta$ is a (order to weak*) homeomorphism, $\beta \rightarrow \gamma(\beta)$, defined by

$$\mu_\beta = T^*S^*\delta_{\gamma(\beta)},$$

is an order isomorphism and homeomorphism, $n \geq 8\|T\|/\epsilon$ and

$$|\|\mu_\beta\| - \|\mu_{\beta'}\|| < \epsilon/(32\|T\|),$$

for all $\beta, \beta' \leq \omega^n$. The family of measures $\{\delta_{\gamma(\beta)} : \beta \leq \omega^n\}$ is the natural basis of the dual of a 1-complemented subspace Z of $C(\omega^\omega)$ isometric to $C(\omega^n)$. Indeed, according to Remark 2.2 it suffices to take

Z as the subspace of $C(\omega^\omega)$ of all functions constant on order intervals $(\gamma(\beta), \gamma(\beta + 1)]$ for $\beta < \omega^n$. Further because

$$\lim_K \|\mu_{\omega^n}|_{[K, \omega] \times \beta\mathbb{N}}\| = 0,$$

and the restriction to $[1, K] \times \beta\mathbb{N}$ is weak*-continuous, we can assume that the support of μ_γ is contained in $[1, K] \times \beta\mathbb{N}$ for all $\gamma \leq \omega^n$. Notice that $[1, K] \times \beta\mathbb{N}$ is homeomorphic to $\beta\mathbb{N}$ so we may replace $[1, K] \times \beta\mathbb{N}$ by $\beta\mathbb{N}$.

In order to simplify notation we can now assume that we have a projection T from $C(\beta\mathbb{N}, X)$ onto a subspace Y isomorphic by an operator S to $C(\omega^n)$ such that

$$\|T^*S^*z\| \geq \epsilon\|z\|,$$

for all $z \in C(\omega^n)^*$, and

$$\|T^*S^*\delta_\beta - T^*S^*\delta_{\beta'}\| < \epsilon/(8\|T\|),$$

for all $\beta, \beta' \leq \omega^n$. Let

$$g_{\omega^n} = S^{-1}(1_{(0, \omega^n]}) \text{ and } g_{\omega^{n-1}k} = S^{-1}(1_{(\omega^{n-1}(k-1), \omega^{n-1}k]}),$$

for all $k \in \mathbb{N}$. ($g_{\omega^{n-1}k}$) is w.u.c. Because X does not contain c_0 , for each t , $(g_{\omega^{n-1}k}(t))$ is unconditionally converging and thus converges in norm to 0 for all $t \in \beta\mathbb{N}$. Because $\|g_{\omega^{n-1}k}(\cdot)\| \leq \|S^{-1}\|$ for all k , $(\|g_{\omega^{n-1}k}(\cdot)\|)$ converges to 0 weakly in $C(\beta\mathbb{N})$. Because

$$\int g_{\omega^{n-1}k} d\mu_{\omega^{n-1}k} = 1,$$

by Lemma 2.6 there exists a subsequence $(\mu_{\omega^{n-1}k})_{k \in M_1}$ and a sequence of disjoint clopen sets $(G_{\omega^{n-1}k})_{k \in M_1}$ such that

$$\int g_{\omega^{n-1}k} 1_{G_{\omega^{n-1}k}} d\mu_{\omega^{n-1}k} \geq 7/8,$$

for all $k \in M_1$. If there is an infinite subset K of M_1 and $\delta > 0$ such that

$$(3.1) \quad \left| \int g_{\omega^n} 1_{G_{\omega^{n-1}k}} d\mu_{\omega^{n-1}k} \right| \geq \delta,$$

for all $k \in K$, then Lemma 3.1 would imply that $C(\omega^\omega)$ is non-separable. Notice that the same contradiction would result if for each k , we replace $G_{\omega^{n-1}k}$ in (3.1) by any of its clopen subsets.

We also have that

$$\int g_{\omega^n} d\mu_{\omega^{n-1}k} = 1,$$

for all k , thus, by replacing M_1 by an infinite subset, for each $k \in M_1$ there are disjoint clopen sets $G_{\omega^{n-1}k}^0$ and $G_{\omega^{n-1}k}^1$ such that

- $\int g_{\omega^{n-1}k} 1_{G_{\omega^{n-1}k}^1} d\mu_{\omega^{n-1}k} > 3/4$,
- $\int g_{\omega^n} 1_G d\mu_{\omega^{n-1}k} < 1/8$ for all clopen $G \subset G_{\omega^{n-1}k}^1$,
- $\int g_{\omega^n} 1_{G_{\omega^{n-1}k}^0} d\mu_{\omega^{n-1}k} > 3/4$.

This is the first step of an at most n -step induction argument.

Fix $k_1 \in M_1$. Consider the sequence $(\omega^{n-1}(k_1 - 1) + \omega^{n-2}k)$. For sufficiently large k ,

$$\int g_{\omega^{n-1}k_1} 1_{G_{\omega^{n-1}k_1}^1} d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} > 3/4,$$

and

$$\int g_{\omega^n} 1_{G_{\omega^{n-1}k_1}^0} d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} > 3/4.$$

Because

$$g_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} = S^{-1}(1_{(\omega^{n-1}(k_1-1)+\omega^{n-2}(k-1), \omega^{n-1}(k_1-1)+\omega^{n-2}k]})$$

converges weakly to 0, by applying Lemma 2.6, there exist a subsequence

$$(\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k})_{k \in M_2}$$

and disjoint clopen sets $(G_k)_{k \in M_2}$ such that

$$\int g_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} > 7/8,$$

for all $k \in M_2$. For every $\delta > 0$ and clopen $H_k \subset G_k$ for $k \in M_2$, Lemma 3.1 tells us that there are only finitely many k for which

$$|\int g_{\omega^{n-1}k_1} 1_{H_k} d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}| > \delta,$$

or

$$|\int g_{\omega^n} 1_{H_k} d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}| > \delta.$$

Thus taking $\delta = 1/8$, for sufficiently large $k \in M_2$ we can find disjoint clopen sets

$$G_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}^j,$$

$j = 0, 1, 2$, such that

- $\int g_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} 1_{G_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}^2} d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} > 5/8$,
- $\int g_{\omega^{n-1}k_1} 1_{G_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}^1} d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} > 5/8$,
- $\int g_{\omega^n} 1_{G_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}^0} d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} > 5/8$.

An induction argument shows that we can choose k_1, k_2, \dots, k_n and disjoint clopen sets

$$G_{\omega^{n-1}(k_1-1)+\omega^{n-2}(k_2-1)+\dots+k_n}^j,$$

$j = 0, 1, \dots, n-1$, such that

$$\int g_{\omega^{n-1}(k_1-1)+\omega^{n-2}(k_2-1)+\dots+\omega^{n-j}k_j} 1_{G_{\omega^{n-1}(k_1-1)+\dots+k_n}^j} d\mu_{\omega^{n-1}(k_1-1)+\dots+k_n}$$

is strictly greater than $1/2 + 1/2^n$. This implies that

$$\|\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}(k_2-1)+\dots+k_n}\| > n/2 > \|T\| \|\delta_{\omega^{n-1}(k_1-1)+\omega^{n-2}(k_2-1)+\dots+k_n}\|.$$

This contradiction shows that no such projection T exists. \square

Remark 3.3. The conclusion of this proposition is equivalent to the statement that $C(\beta\mathbb{N} \times \omega, X)$ does not contain $C(\omega^n)$ uniformly complemented. Obviously if X contains $C(\omega^n)$ uniformly complemented then it follows that $C(\beta\mathbb{N} \times \omega, X)$ contains $C(\omega^n)$ uniformly complemented. It is possible that the hypothesis on X could be weakened to something like X does not contain $C(\omega^n)$ uniformly or uniformly complemented. We do not know whether $C(\beta\mathbb{N} \times \omega, C(\beta\mathbb{N}))$ contains a complemented subspace isomorphic to $C(\omega^\omega)$. If this is a counterexample then assuming additionally that X is separable may provide a strong enough hypothesis.

The next result generalizes the previous to larger ordinals.

Theorem 3.4. *Let X be a Banach space and $0 \leq \alpha < \beta < \omega_1$. Then*

$$C(\omega^{\omega^\beta}) \xhookrightarrow{c} C(\beta\mathbb{N} \times \omega^{\omega^\alpha}, X) \implies c_0 \hookrightarrow X.$$

Proof. Let α be a countable ordinal and X a Banach space containing no copy of c_0 . We will show by induction that α is the smallest ordinal γ such that $C(\beta\mathbb{N} \times \omega^{\omega^\gamma}, X)$ contains a complemented subspace isomorphic to $C(\omega^{\omega^\alpha})$. Theorem 3.2 shows that for $\alpha = 1$ this is the case. Assume that the result holds for ordinals less than α , $\alpha > 1$, and that $\gamma < \alpha$ is the smallest ordinal such that $C(\beta\mathbb{N} \times \omega^{\omega^\gamma}, X)$ contains a complemented subspace isomorphic to $C(\omega^{\omega^\alpha})$. We will show that this leads to a contradiction.

In place of $C(\beta\mathbb{N} \times \omega^{\omega^\gamma}, X)$, we will use the isomorphic space, $C_0(\omega^{\omega^\gamma} \times \beta\mathbb{N}, X)$. Now assume that T is a projection defined on $C_0(\omega^{\omega^\gamma} \times \beta\mathbb{N}, X)$ with range isomorphic to $C(\omega^{\omega^\alpha})$. Let $\alpha_k \uparrow \omega^\alpha$ and $\beta_k \uparrow \omega^\gamma$, where $\alpha_k = \omega^{\alpha'} k$ if $\alpha = \alpha' + 1$ for some α' , or $\alpha_k = \omega^{\xi_k}$ if α is a limit ordinal and $\xi_k \uparrow \alpha$, and $\beta_k = \omega^{\gamma'} k$ if $\gamma = \gamma' + 1$ for some γ' , or $\beta_k = \omega^{\gamma_k}$ if γ is a limit ordinal and $\gamma_k \uparrow \gamma$. Choose k_0 such that $\alpha_k \geq \omega^\gamma$ for all $k \geq k_0$. By Lemma 2.5 for each $k > k_0$, there is a k' such that $\{T^* \delta_\beta : \beta \leq \omega^{\alpha_{k'}}\}$ contains a subfamily $\{\mu_\rho : \rho \leq \omega^{\alpha_k}\}$ such that $\rho \rightarrow \mu_\rho$ is a homeomorphism, $\rho \rightarrow \beta(\rho)$ is an order homeomorphism, where $\mu_\rho = T^* \delta_{\beta(\rho)}$, and

$$\|\mu_\rho\| - \|\mu_{\rho'}\| < 1/(4\|T\|),$$

for all $\{\rho, \rho' \leq \omega^{\alpha_k}\}$. For each m let P_m be the canonical projection from $C_0(\omega^{\omega^\gamma} \times \beta\mathbb{N}, X)$ onto $C(\omega^{\beta_m} \times \beta\mathbb{N}, X)$. There exists an m such that

$$\|(I - P_m^*)(\mu_{\omega^{\alpha_k}})\| < 1/(8\|T\|).$$

It follows by passing to a suitable neighborhood of ω^{α_k} that we may assume that

$$\|(I - P_m^*)(\mu_\rho)\| < 3/(8\|T\|),$$

for all $\rho \leq \omega^{\alpha_k}$. According to Remark 2.2 we can find a 1-complemented subspace Z of $C(\omega^{\omega^\alpha})$ which is isometric to $C(\omega^{\alpha_k})$ and has natural basis of its dual $\{\mu_\rho : \rho \leq \omega^{\alpha_k}\}$. This implies that $P_m(Z)$ is a complemented subspace of $C(\beta\mathbb{N} \times \omega^{\beta_m}, X)$ isomorphic to $C(\omega^{\alpha_k})$. Because $\beta_m < \omega^\gamma$ and $\alpha_k \geq \omega^\gamma$, $C(\beta\mathbb{N} \times \omega^{\beta_m}, X)$ cannot contain a complemented copy of $C(\omega^{\omega^\gamma})$ by the inductive hypothesis. Thus we have a contradiction and the theorem is proved. \square

Now we can prove

Theorem 3.5. *Let X be a Banach space containing no copy of c_0 , K an infinite compact metric space and $0 \leq \alpha < \omega_1$. Then*

$$C(K) \xhookrightarrow{c} C(\beta\mathbb{N} \times \omega^{\omega^\alpha}, X) \iff C(K) \sim C(\omega^{\omega^\xi}) \text{ for some } 0 \leq \xi \leq \alpha.$$

Proof. Since that $C([0, 1])$ contains complemented copies of every $C(\omega^{\omega^\alpha})$, $0 \leq \alpha < \omega_1$, it follows directly from Milutin's theorem and Theorem 3.4 that K must be countable if $C(K) \xhookrightarrow{c} C(\beta\mathbb{N} \times \omega^{\omega^\alpha}, X)$. If K is countable, $C(K)$ is isomorphic to $C(\omega^{\omega^\xi})$ for some countable ordinal ξ and Theorem 3.4 determines the possible values of ξ . The converse is obvious. \square

Remark 3.6. Because $C(\beta\mathbb{N}, l_1)$ is isomorphic to its c_0 -sum, see Theorem 5.4, the above result in the case $X = l_1$ does not mimic that for the scalar case where there is an additional isomorphism class. Indeed, since c_0 is not isomorphic to a complemented subspace of $C(\beta\mathbb{N})$, $C(\beta\mathbb{N})$ and $C(\beta\mathbb{N} \times \omega)$ are not isomorphic.

We can now prove the main result of this section.

Theorem 3.7. *Let X be a Banach space containing no copy of c_0 . Then for any infinite compact metric spaces K_1 and K_2 we have*

$$C(\beta\mathbb{N} \times K_1, X) \sim C(\beta\mathbb{N} \times K_2, X) \iff C(K_1) \sim C(K_2).$$

Proof. Let us show the non trivial implication. Suppose that

$$C(\beta\mathbb{N} \times K_1, X) \sim C(\beta\mathbb{N} \times K_2, X).$$

It is convenient to consider two subcases:

Case 1. K_1 and K_2 are countable. Hence there are countable ordinals ξ and η such that $C(K_1) \sim C(\omega^{\omega^\xi})$ and $C(K_2) \sim C(\omega^{\omega^\eta})$. Then according to our hypothesis,

$$C(\omega^{\omega^\eta}) \xrightarrow{c} C(\omega^{\omega^\eta} \times \beta\mathbb{N}, X) \sim C(\omega^{\omega^\xi} \times \beta\mathbb{N}, X).$$

Therefore by Theorem 3.5. we deduce that $\omega^{\omega^\eta} \leq \omega^{\omega^\xi}$. Similarly, we show that $\omega^{\omega^\xi} \leq \omega^{\omega^\eta}$. Hence $C(K_1) \sim C(K_2)$.

Case 2. K_1 or K_2 is uncountable. Without loss of generality we suppose that K_2 is uncountable. To prove that $C(K_1) \sim C(K_2)$, it is enough by Milutin's theorem to show that K_1 is also uncountable. Assume to the contrary. So there exists an ordinal ξ such that $C(K_1) \sim C(\omega^{\omega^\xi})$. Since $C(K_2) \sim C([0, 1])$ we have by our hypothesis that

$$C(\omega^{\omega^{\xi+1}}) \xrightarrow{c} C(\beta\mathbb{N} \times [0, 1], X) \xrightarrow{c} C(\omega^{\omega^\xi} \times \beta\mathbb{N}, X),$$

a contradiction of Theorem 3.5. This completes the proof of Theorem 3.7. □

Corollary 3.8. *Let X be a Banach space containing no copy of c_0 . Then for any infinite compact metric spaces K_1 and K_2 we have*

$$C(K_1, X) \sim C(K_2, X) \iff C(K_1) \sim C(K_2)$$

Proof. One direction is immediate. If $C(K_1, X) \sim C(K_2, X)$ then $C(\beta\mathbb{N} \times K_1, X) \sim C(\beta\mathbb{N} \times K_2, X)$, so this follows from the previous result. □

The next result can be considered as an extension of the Cembranos-Freniche result however the proof does not yield a proof of that result. To include the original we would need to use the Josefson-Nissenzweig Theorem, [17] and [23].

Proposition 3.9. *Suppose that $0 \leq \alpha < \omega_1$, $0 \leq \gamma < \omega_1$, and (γ_n) is either (ω^{β_n}) where (β_n) increases to γ or (γ_n) is $(\omega^\beta n)$ and $\gamma = \beta + 1$ for some ordinal β . X is a Banach space such that with constants independent of n , $C(\omega^{\gamma_n})$ is isomorphic to a complemented subspace of X . Then for any infinite compact Hausdorff space K , $C(\omega^{\omega^\alpha} \times \omega^{\omega^\gamma})$ is isomorphic to a complemented subspace of $C(K \times \omega^{\omega^\alpha}, X)$.*

If X is also separable then $C(\omega^{\omega^\alpha})$ is isomorphic to a complemented subspace of $C(K, X)$.

Proof. Clearly $C(\omega^{\omega^\alpha})$ is isomorphic to a complemented subspace of $C(K \times \omega^{\omega^\alpha}, X)$. We also know by Lemma 2.4 that

$$C(\omega^{\omega^\alpha} \times \omega^{\omega^\gamma}) \sim C(\omega^{\omega^{\max\{\alpha, \gamma\}}}).$$

So we need only show that $C(\omega^{\omega^\gamma})$ is isomorphic to a complemented subspace of $C(K \times \omega^{\omega^\alpha}, X)$. The case $\gamma = 0$ is the Cembranos-Freniche result but also is immediate from the fact that $\alpha \geq 0$. Now assume $\gamma \geq 1$.

Notice that $C(K \times \omega^{\omega^\alpha}, X)$ is isomorphic to $C_0(\omega \times \omega^{\omega^\alpha} \times K, X)$. This in turn is isomorphic to

$$(\sum_{j \in \mathbb{N}} C(\omega^{\omega^\alpha} \times K, X))_{c_0}.$$

For each $n \in \mathbb{N}$ let X_n be a complemented subspace of X which is isomorphic to $C(\omega^{\gamma_n})$ and let P_n be a projection from X onto X_n . By the hypothesis we can assume that the norms of the isomorphisms and the projections are bounded independent of n . Choose any point $a \in \omega^{\omega^\alpha} \times K$. If $(f_j) \in (\sum_j C(\omega^{\omega^\alpha} \times K, X))_{c_0}$, then

$$P((f_j)) = (P_j(f_j(a))1_{\omega^{\omega^\alpha} \times K})$$

defines a projection onto a space isometric to the c_0 -sum of X_j . Because $(\sum_{j \in \mathbb{N}} C(\omega^{\gamma_j}))_{c_0}$ is isomorphic to $C(\omega^{\omega^\gamma})$, the c_0 -sum of X_j is isomorphic to $C(\omega^{\omega^\gamma})$.

If X is separable, then with X_n as before let Y_n be the subspace of X_n which is the image of $C_0(\omega^{\gamma_n})$ under the isomorphism from $C(\omega^{\gamma_n})$ and Q_n be the projection from X onto Y_n . Because X is separable there is a decreasing sequence of weak*-open sets G_j which is a base for the neighborhood system of 0 in the ball of X^* , B_{X^*} . For each n there is a sequence of complemented subspaces $Y_{n,k}$ of Y_n with projections $Q_{n,k}$, $Y_{n,k} \supset Y_{n,k+1}$ and $Y_{n,k}$ is isomorphic to $C_0(\omega^{\gamma_n})$ for all k , such that for each j and n there is a K such that for all $k \geq K$,

$$Q_{n,k}^*(X^*) \cap B_{X^*} \subset G_j.$$

Indeed if $\rho_k \nearrow \omega^{\gamma_n}$, then $f \rightarrow f1_{(\rho_k, \omega^{\gamma_n}]}$ is a projection onto a subspace of $C_0(\omega^{\gamma_n})$ isomorphic to $C_0(\omega^{\gamma_n})$ and $Y_{n,k}$ can be taken to be the image of this subspace in Y_n .

Let (g_n) be a sequence of disjointly supported non-negative norm one elements in $C(K)$ such that for each n there is an open set

$$H_n \supset \overline{\{t : g_n(t) > 0\}},$$

with $H_n \cap H_m = \emptyset$ for all $m \neq n$, and $t_n \in K$ such that $g_n(t_n) = 1$ for all n . Let

$$D = \sup_{s,k} \|Q_{s,k}\|,$$

and choose k_n such that

$$D^{-1}Q_{n,k_n}^*(B_{X^*}) \subset G_n,$$

for all n and let

$$Z = [g_n y_n : y_n \in Y_{n,k_n}, n \in \mathbb{N}].$$

Clearly Z is isomorphic to $(\sum_n C_0(\omega^{\gamma_n}))_{c_0}$ which is isomorphic to $C(\omega^{\omega^\gamma})$. Define an operator T from $C(K, X)$ into Z by

$$Tf(t) = g_n(t)Q_{n,k_n}(f(t_n)),$$

for all $t \in \text{supp } g_n$ and $Tf(t) = 0$ if $t \notin \cup_n \text{supp } g_n$. Because each g_n is continuous, Tf is continuous on H_n for all n . If $\epsilon > 0$, $t'_n \in \text{supp } g_n$, and t is an accumulation point of $\{t'_n\}$, by the continuity of f , $\|f(t) - f(t'_n)\| < \epsilon$ for all $t'_n \in H$ where H is some neighborhood of t . Because t cannot be in any H_n , $T(f(t)) = 0$. By the choice of k_n we have that

$$\lim_n \sup_{x^* \in B_{X^*}} |(Q_{n,k_n}^* x^*)(f(t))| = 0$$

and for $t'_n \in H$,

$$\epsilon \sup_{s,k \in \mathbb{N}} \|Q_{s,k}\| > \sup_{x^* \in B_{X^*}} |(Q_{n,k_n}^* x^*)(f(t'_n)) - (Q_{n,k_n}^* x^*)(f(t))|.$$

Thus

$$\|(Tf)(t)\| = 0 = \lim_{n \in \mathcal{N}} \sup_{x^* \in B_{X^*}} |(Q_{n,k_n}^* x^*)(f(t_n))g(t'_n)| = \lim_{n \in \mathcal{N}} \|(Tf)(t'_n)\|,$$

where the limit is over some net $(t'_n)_{n \in \mathcal{N}}$ so that $\lim_{n \in \mathcal{N}} t'_n = t$.

It is easy to see that

$$\|T\| \leq \sup_{s,k \in \mathbb{N}} \|Q_{s,k}\|$$

and, because $g_n(t_n) = 1$ and each Q_{n,k_n} is a projection, that T is a projection. \square

Remark 3.10. We do not whether the separability condition in the second part is necessary but the argument fails for the natural choices of X_n if $X = \{(x_n) : \forall n \in \mathbb{N}, x_n \in C(\omega^n), \|(x_n)\| = \sup_n \|x_n\| < \infty\}$.

In the next section we will prove some results about quotients of $C(K, X)$ isomorphic to $C(\omega^{\omega^\alpha})$. If we consider quotients in the previous proposition instead of complemented subspaces, the analogous results hold. The proof is similar to the previous one except that the argument is now entirely in the dual.

Proposition 3.11. *Suppose that $0 \leq \alpha < \omega_1$, $0 \leq \gamma < \omega_1$, and (γ_n) is either (ω^{β_n}) where (β_n) increases to γ or (ω^{β_n}) and $\gamma = \beta + 1$. X is a Banach space such that with constants independent of n , $C(\omega^{\gamma_n})$ is isomorphic to a quotient of X . Then for any infinite compact Hausdorff space K , $C(\omega^{\omega^\alpha} \times \omega^{\omega^\gamma})$ is isomorphic to a quotient of $C(K \times \omega^{\omega^\alpha}, X)$.*

If X is also separable then $C(\omega^{\omega^\gamma})$ is isomorphic to a quotient of $C(K, X)$.

Proof. Clearly $C(\omega^{\omega^\alpha})$ is isomorphic to a quotient of $C(K \times \omega^{\omega^\alpha}, X)$. We also know by Lemma 2.4 that

$$C(\omega^{\omega^\alpha} \times \omega^{\omega^\gamma}) \sim C(\omega^{\max\{\alpha, \gamma\}}).$$

Thus as before we need only show that $C(\omega^{\omega^\gamma})$ is isomorphic to a quotient of $C(K \times \omega^{\omega^\alpha}, X)$. The case $\gamma = 0$ is the Cembranos-Freniche result but also is immediate from the fact that $\alpha \geq 0$. Assume $\gamma \geq 1$.

Notice that $C(K \times \omega^{\omega^\alpha}, X)$ is isomorphic to $C_0(\omega \times \omega^{\omega^\alpha} \times K, X)$ and this is isomorphic to

$$(\sum_j C(\omega^{\omega^\alpha} \times K, X))_{c_0}.$$

For each $n \in \mathbb{N}$ let X_n be a quotient of X which is isomorphic to $C(\omega^{\gamma^n})$. P_n be a quotient map from X onto X_n . By the hypothesis we can assume that the norms of the isomorphisms and the quotient maps are bounded independent of n . Thus $P_n^*(X_n^*)$ is weak*-isomorphic to $C(\omega^{\gamma^n})^*$ and the subspace

$$Z = \{(z_n) : z_n \in P_n^*(X_n^*) \text{ for all } n \in \mathbb{N}\}$$

of $(\sum_j C(\omega^{\omega^\alpha} \times K, X))_{c_0}^*$ is weak*-isomorphic to $((\sum_j C(\omega^{\gamma^n}))_{c_0})^*$. This space is weak*-isomorphic to $C(\omega^\gamma)^*$ giving us the required quotient.

If X is separable, then with X_n as before let Y_n be the complemented subspace of X_n which is the image of $C_0(\omega^{\gamma^n})$ under the isomorphism and Q_n be the quotient map from X onto Y_n . Because X is separable there is a decreasing sequence of weak*-open sets G_j which is a base for the neighborhood system of 0 in the ball of X^* , B_{X^*} . As in the proof of Proposition 3.9 for each n there is a sequence of complemented subspaces $Y_{n,k}$ of Y_n with projections $Q_{n,k}$, $Y_{n,k} \supset Y_{n,k+1}$ and $Y_{n,k}$ is isomorphic to $C_0(\omega^{\gamma^n})$ for all k , such that for each j and n there is a K such that for all $k \geq K$,

$$Q_{n,k}^*(Y_n^*) \cap B_{X^*} \subset G_j.$$

Let (t_n) be a sequence of points in K such that for each n there is an open set H_n containing t_n with $H_n \cap H_m = \emptyset$ for all $m \neq n$, for all n . Let

$$D = \sup_{s,k} \|Q_{s,k}\|,$$

and choose k_n such that

$$D^{-1}Q_{n,k_n}^*(B_{Y_n^*}) \subset G_n,$$

for all n and let

$$Z = [z_n \delta_{t_n} : z_n \in Q_{n,k_n}^*(Y_n^*) \text{ for all } n \in \mathbb{N}].$$

Clearly Z is isomorphic to $((\sum_n C_0(\omega^{\gamma_n})^*)_{\ell_1})^*$. We need to show that Z is w^* -isomorphic to $((\sum_n C_0(\omega^{\gamma_n}))_{c_0})^*$. This however follows immediately from the choice of (k_n) and the fact that no point t_j is an accumulation point of $\{t_n : n \in \mathbb{N}\}$. (As was shown in the proof of Proposition 3.9). \square

4. SEPARABLE $C(K)$ QUOTIENTS OF $C(\beta\mathbb{N} \times \alpha, X)$

By Theorem 1.2 we know that $C(\beta\mathbb{N}, l_p)$, $1 < p < \infty$, contains a complemented copy of c_0 . The main aim of this section is to show that $C(\omega^\omega)$ is not even a quotient of this space, (Proposition 4.3). Of course, this implies that c_0 is, up to an isomorphism, the only separable $C(K)$ space which is a quotient of $C(\beta\mathbb{N}, l_p)$, $1 < p < \infty$.

In this section we will work with Banach spaces X that satisfy the following properties.

- (\dagger) X^* has a monotone weak*-FDD (X_m^*) .
- (\ddagger) For every constant C , $0 < C < 1$, there is a constant C' such that for all $x^* \in X^*$ and $j \in \mathbb{N}$,

$$\|(I - P_j)x^*\| \leq C\|x^*\| + C'(\|x^*\| - \|P_j x^*\|),$$

where (P_j) is the sequence of FDD projections, i.e., $P_j(X^*) = [X_m^* : m \leq j]$.

We will refer to such spaces as *satisfying the daggers*. Before proceeding to the main results we will verify that ℓ_p , for $1 < p < \infty$, satisfies the daggers.

The following lemma follows from the Mean Value Theorem.

Lemma 4.1. *Suppose that $0 < C < 1$, $1 < q < \infty$, $g(t) = (1 - t^q)^{1/q}$ and t_0 satisfies $g(t_0) = C$. Then there is a positive constant C_1 which depends on q such that for all $t_0 > t > 0$*

$$g(t) \leq C + C_1(t_0 - t).$$

If we consider $\ell_p^* = \ell_q$ with the standard basis and basis projections, then we have that for $x^* \in \ell_q$, $x^* \neq 0$,

$$\|x^*\| = (\|P_j x^*\|^q + \|x^* - P_j x^*\|^q)^{1/q}.$$

Let $\lambda = \|P_j x^*\|/\|x^*\|$ and rewriting we have that

$$\|x^* - P_j x^*\| = \|x^*\|(1 - \lambda^q)^{1/q}.$$

If C is given and $\lambda \leq t_0$ where t_0 and C_1 are as in the lemma,

$$\|x^* - P_j x^*\| \leq C\|x^*\| + C_1\|x^*\|(t_0 - \lambda) \leq C\|x^*\| + C_1(\|x^*\| - \|P_j x^*\|).$$

If $\lambda > t_0$, then $(1 - \lambda^q)^{1/q} < C$ and the first term suffices. Thus we see that

Corollary 4.2. ℓ_p , $1 < p < \infty$, satisfies the daggers.

The next result is the initial case of the main theorem of this section.

Theorem 4.3. *Suppose that X is a Banach space satisfying the daggers. Then $C(\omega^\omega)$ is not a quotient of $C(\omega \times \beta\mathbb{N}, X)$.*

Proof. First we can replace $C(\omega \times \beta\mathbb{N}, X)$ by the isomorphic space $C_0(\omega \times \beta\mathbb{N}, X)$. Suppose that there exists a bounded linear operator T from $C_0(\omega \times \beta\mathbb{N}, X)$ onto $C(\omega^\omega)$. We will show that this leads to a contradiction. We may assume that $\|T\| = 1$. Because T^* is a weak*-isomorphism from $C(\omega^\omega)^*$ into $C_0(\omega \times \beta\mathbb{N}, X)^*$, there is a constant K such that for every $n \in \mathbb{N}$, $\{\delta_\alpha : \alpha \leq \omega^n\}$ is mapped to $\{x_\alpha^* : \alpha \leq \omega^n\} \subset C_0(\omega \times \beta\mathbb{N}, X)^*$ and $\{x_\alpha^* : \alpha \leq \omega^n\}$ is K -equivalent to the usual unit vector basis of ℓ_1 , i.e.,

$$\left\| \sum_{\alpha} c_{\alpha} x_{\alpha}^* \right\| \geq \sum_{\alpha} |c_{\alpha}| / K,$$

for all sequences of scalars (c_{α}) .

Let $C = (8K)^{-1}$ in (\ddagger) , and choose ρ , $1/2 > \rho > 0$, such that

$$\left((1 - \rho)^{-1} \left(1 - \frac{(1 - \rho)^3}{1 + \rho} \right) \right) C' < (4K)^{-1}.$$

By Lemma 2.5 for $k = 1$ and n sufficiently large, we can find $(x_{\alpha(\gamma)}^*)_{\gamma \leq \omega}$ with

$$|\|x_{\alpha(\gamma)}^*\| - \|x_{\alpha(\gamma')}^*\|| < \rho/K,$$

for all $\gamma, \gamma' \leq \omega$. Moreover, as in the proof of Theorem 3.2, we may assume that the measures are all supported in $[1, K] \times \beta\mathbb{N}$ and thus reduce to measures supported on $\beta\mathbb{N}$. By our identification of $C(\beta\mathbb{N}, X)^*$ with a space of X^* -valued measures and switching to a more suggestive notation we have (μ_n) , a weak* converging sequence of X^* -valued measures with limit μ , with

$$\|\mu\|(1 + \rho) > \|\mu_n\| > \|\mu\|(1 - \rho),$$

for all n .

Choose a finite partition $\{B_j\}$ of $\beta\mathbb{N}$ into clopen sets such that

$$\sum_j \|\mu(B_j)\|_{X^*} > \|\mu\|(1 - \rho).$$

As in (\dagger) let P_m denote the FDD projection of X^* onto the span of the first m subspaces. In a slight abuse of notation we will also use P_m for the operator on the X^* -valued measures defined by

$$(P_m\mu)(A) = P_m(\mu(A)),$$

for all measurable A . Choose N such that

$$\|P_N(\mu(B_j))\| > (1 - \rho)\|\mu(B_j)\|,$$

for all j . By passing to a subsequence we may assume that

$$\|P_N(\mu_n(B_j))\| > (1 - \rho)\|\mu(B_j)\|,$$

for all j and n . Hence

$$\begin{aligned} \|P_N\mu_n\| &\geq \sum_j \|P_N\mu_n(B_j)\| \geq (1 - \rho) \sum_j \|\mu(B_j)\| \\ &\geq (1 - \rho)^2 \|\mu\| \geq \frac{(1 - \rho)^2}{1 + \rho} \|\mu_n\|. \end{aligned}$$

Fix n and choose a partition $\{A_k\}$ that refines $\{B_j\}$, such that

- $\sum_k \|\mu_n(A_k)\| \geq (1 - \rho)\|\mu_n\|$,
- $\sum_k \|P_N\mu_n(A_k)\| \geq (1 - \rho)\|P_N\mu_n\|$,
- $\sum_k \|(I - P_N)\mu_n(A_k)\| \geq (1 - \rho)\|(I - P_N)\mu_n\|$.

For each k let λ_k satisfy

$$\lambda_k \|\mu_n(A_k)\| = \|P_N\mu_n(A_k)\|.$$

Then

$$\sum_k \lambda_k \|\mu_n(A_k)\| \geq \frac{(1 - \rho)^3}{1 + \rho} \|\mu_n\| \geq \frac{(1 - \rho)^3}{1 + \rho} \sum_k \|\mu_n(A_k)\|.$$

Equivalently,

$$(1 - \frac{(1 - \rho)^3}{1 + \rho}) \sum_k \|\mu_n(A_k)\| \geq \sum_k (1 - \lambda_k) \|\mu_n(A_k)\|.$$

We need to estimate $\|(I - P_N)\mu_n\|$. Let

$$M = \{k : \|(I - P_k)\mu_n(A_k)\| \leq C\|\mu_n(A_k)\|\}.$$

Then by the choice of C , $\sum_k \|(I - P_N)\mu_n(A_k)\|$ is less than or equal to

$$\begin{aligned} & \sum_{k \in M} C\|\mu_n(A_k)\| + \sum_{k \notin M} (C + C'(1 - \lambda_k))\|\mu_n(A_k)\| \\ & \leq \sum_k C\|\mu_n(A_k)\| + C' \sum_k (1 - \lambda_k)\|\mu_n(A_k)\| \\ & \leq C\|\mu_n\| + C'(1 - \frac{(1 - \rho)^3}{1 + \rho})\|\mu_n\|. \end{aligned}$$

Because $\|T\| = 1$, $\|\mu_n\| \leq 1$. Therefore $\|(I - P_N)\mu_n\|$ is less than or equal to

$$(1 - \rho)^{-1}C + (1 - \rho)^{-1}C'(1 - \frac{(1 - \rho)^3}{1 + \rho}) \leq (2K)^{-1}.$$

Notice that $(P_N\mu_n)$ converges to $P_N\mu$ in the weak*-topology. Because $C(\beta\mathbb{N})$ is a Grothendieck space so is $C(\beta\mathbb{N}, [X_m : m \leq N])$. Thus $(P_N\mu_n)$ converges to $P_N\mu$ in the weak topology. Because (μ_n) is K -equivalent to the usual unit vector basis of ℓ_1 , the estimate on $\|(I - P_N)\mu_n\|$ implies that $(P_N\mu_n)$ is also equivalent to the usual unit vector basis of ℓ_1 . This is a contradiction because the unit vector basis of ℓ_1 has no weak Cauchy subsequence. \square

Remark 4.4. The proof of the theorem shows that $C(\omega^n)$ is not a quotient of $C(\omega \times \beta\mathbb{N}, X)$ uniformly in n . Because $C(\omega^\omega)$ is isomorphic to $(\sum_n C(\omega^n))_{c_0}$, this is equivalent to $C(\omega^\omega)$ not being a quotient. It is conceivable that the conclusion of Theorem 4.3 could be proved under the hypothesis that X does not have $C(\omega^n)$ as a quotient uniformly in n .

The following theorem generalizes Theorem 4.3 to higher ordinals.

Theorem 4.5. *Suppose that X is a Banach space such that $C(\omega^\omega)$ is not isomorphic to a quotient of $C(\omega, X)$. Let $\alpha \geq 1$. Then $C(\omega^{\omega^\alpha})$ is isomorphic to a quotient of $C(\beta\mathbb{N} \times \xi, X)$ if and only if $\xi \geq \omega^{\omega^\alpha}$.*

Proof. It is easy to see that $C(\gamma)$ is isomorphic to a complemented subspace of $C(\beta\mathbb{N} \times \gamma, X)$ for all $\gamma \geq 1$. Thus the sufficiency is clear.

On the other hand, by Proposition 2.1

$$C(\beta\mathbb{N} \times \omega^n, X) = C(\omega^n, C(\beta\mathbb{N}, X)) \sim C(\omega, C(\beta\mathbb{N}, X)) = C(\beta\mathbb{N} \times \omega, X),$$

for all positive integers n . Thus the necessity is true for $\alpha = 1$. Because we have that $C(\omega^\omega)$ is not isomorphic to a quotient of $C(\omega, X)$ or of $C(\beta\mathbb{N} \times \omega, X)$, it is sufficient to prove the result for $C(\xi, X)$ rather than $C(\beta\mathbb{N} \times \xi, X)$.

Now suppose that $\alpha > 1$ and for all $\gamma < \alpha$, $C(\omega^{\omega^\gamma})$ is not isomorphic to a quotient of $C(\xi, X)$ if $\xi < \omega^{\omega^\gamma}$. We will show that if $\beta < \omega^{\omega^\alpha}$ and there is a bounded operator T from $C(\beta, X)$ onto $C(\omega^{\omega^\alpha})$ that this leads to a contradiction. Without loss of generality we assume that $\|T\| = 1$.

If α is a limit ordinal and $\alpha_n \uparrow \alpha$, then $\omega^{\omega^{\alpha_n}} > \beta$ for some n . $C(\omega^{\omega^{\alpha_n}})$ is isomorphic to a quotient of $C(\omega^{\omega^\alpha})$ and by the assumption also $C(\beta, X)$ but this contradicts the induction hypothesis.

Now suppose that $\alpha = \gamma + 1$, for some ordinal γ . By Proposition 2.1, $C(\beta, X)$ is isomorphic to $C_0(\beta, X)$. Hence we may assume that T goes from $C_0(\beta, X)$ onto $C(\omega^{\omega^\alpha})$. Also by Proposition 2.1 we may assume that $\beta = \omega^{\omega^\zeta}$ for some $1 \leq \zeta < \alpha$. Let $\zeta_k = \omega^{\zeta-1}k$ if ζ is not a limit ordinal and $\zeta_k \uparrow \omega^\zeta$ otherwise. Let $\{x_\rho^* : \rho \leq \omega^{\omega^\alpha}\}$ be the corresponding images of the dirac measures $\{\delta_\rho : \rho \leq \omega^{\omega^\alpha}\}$ under T^* . Then $\{x_\rho^* : \rho \leq \omega^{\omega^\alpha}\}$ is, for some $R > 1$, R -equivalent to the usual unit vector basis of ℓ_1 . Employing Lemma 2.5 there is an N sufficiently large such that if $\{y_\rho^* : \rho \leq \omega^{\omega^{\gamma N}}\}$ is contained in the unit ball of $C_0(\beta, X)^*$ and the mapping $\rho \rightarrow y_\rho^*$ is weak* continuous, then there is a continuous map ϕ from $[1, \omega^{\omega^\gamma}]$ into $[1, \omega^{\omega^{\gamma N}}]$ such that

$$\|y_{\phi(\rho)}^* - y_{\phi(\rho')}^*\| < \frac{1}{4R},$$

for all $\rho, \rho' \leq \omega^{\omega^\gamma}$. Applying this to $\{x_\rho^* : \rho \leq \omega^{\omega^{\gamma N}}\}$, we get a map ϕ as above. Because $C_0(\beta, X)^*$ is weak*-isomorphic, norm-isometric to $(\sum C(\omega^{\zeta_k}, X)^*)_{c_0^*}$, there exists K such that

$$\|(I - P_K)x_{\phi(\omega^{\omega^\gamma})}^*\| < \frac{1}{4R},$$

where P_K is the weak*-continuous projection (truncation) from

$$\left(\sum_{k=1}^{\infty} (C(\omega^{\zeta_k}, X)^*)_{c_0^*}\right) \text{ onto } \left(\sum_{k \leq K} (C(\omega^{\zeta_k}, X)^*)_{c_0^*}\right).$$

By passing to a neighborhood of ω^{ω^γ} we may assume that

$$\|P_K x_{\phi(\rho)}^*\| > \|P_K x_{\phi(\omega^{\omega^\gamma})}^*\| - \frac{1}{4R} > \|x_{\phi(\rho)}^*\| - \frac{3}{4R},$$

for all $\rho \leq \omega^{\omega^\gamma}$. Thus $C(\omega^{\omega^\gamma})^*$ is weak*-isomorphic to a subspace of

$$\left(\sum_{k \leq K} C(\omega^{\zeta_k}, X)^*\right)_{c_0^*},$$

and consequently $C(\omega^{\omega^\gamma})$ is isomorphic to a quotient of

$$C(\omega^{\zeta_1} + \cdots + \omega^{\zeta_K}, X).$$

But $\omega^{\zeta_1} + \dots + \omega^{\zeta_K} < \omega^{\omega^\gamma}$ contradicting the inductive hypothesis. No such β exists. \square

The purpose of this section is to prove Theorem 4.6. This result now follows from Theorem 4.5 and Milutin's theorem by an argument similar to the deduction of Theorem 3.5 from Theorem 3.2. We leave the details to the reader.

Theorem 4.6. *Let K be an infinite compact metric space and $0 \leq \alpha < \omega_1$, and let X satisfy the daggers. Then*

$$C(\beta\mathbb{N} \times \omega^{\omega^\alpha}, X) \twoheadrightarrow C(K) \iff C(K) \sim C(\omega^{\omega^\xi}) \text{ for some } 0 \leq \xi \leq \alpha.$$

Corollary 4.7. *Let K be an infinite compact metric space and $0 \leq \alpha < \omega_1$. Then*

$$C(\beta\mathbb{N} \times \omega^{\omega^\alpha}, l_p) \twoheadrightarrow C(K) \iff C(K) \sim C(\omega^{\omega^\xi}) \text{ for some } 0 \leq \xi \leq \alpha.$$

5. THE ISOMORPHISM OF $C(\beta\mathbb{N} \times \omega, l_p)$ AND $C(\beta\mathbb{N}, l_p)$, $1 \leq p < \infty$

As an immediate consequence of Theorem 3.2, for every $1 \leq p < \infty$ and $\alpha \geq \omega^\omega$, we have

$$C(\beta\mathbb{N} \times \alpha, l_p) \not\sim C(\beta\mathbb{N}, l_p).$$

For α finite $\beta\mathbb{N} \times \alpha$ is homeomorphic to $\beta\mathbb{N}$, so the case $\omega \leq \alpha < \omega^\omega$ of Theorem 5.4 remains. This result is the case $p = 1$ of Theorem 5.3. To prove this theorem we need the following lemma.

Lemma 5.1. *Let X and Y be Banach spaces and let $1 \leq p < \infty$ and $p \leq q \leq \infty$ with $(p, q) \neq (1, \infty)$. Then $c_0(\omega \times \mathcal{K}(X, Y))$ is isomorphic to a complemented subspace of $\mathcal{K}(l_p(X), l_q(Y))$.*

Proof. Let $T \in \mathcal{K}(l_p(X), l_q(Y))$. Represent T as a matrix with entries in $\mathcal{K}(X, Y)$. For the moment assume that $q < \infty$. By [20, Proposition 1.c.8 and following Remarks] the operator given by the diagonal of the matrix is a bounded linear operator with norm no larger than $\|T\|$. Therefore the mapping from $\mathcal{K}(l_p(X), l_q(Y))$ into the diagonal operators with respect to this representation is a contraction. If $p > 1$ and $q = \infty$, then sign change operators are contractive and the argument from [20] shows that the map from $\mathcal{K}(l_p(X), l_\infty(Y))$ into the diagonal operators is contractive. Also for compact operators that are diagonal the map is the identity.

The norm of a diagonal operator is the supremum of the norms of the operators on the diagonal. For the case $q = \infty$ this is clear. If $q < \infty$,

let D_j be the j th block of the diagonal operator D and $(x_j) \in l_p(X)$. Then

$$\begin{aligned} \|(D_j x_j)\|_{l_q(Y)} &= \left(\sum \|D_j x_j\|_Y^q \right)^{1/q} \leq \left(\sum \|D_j\|^q \|x_j\|_X^q \right)^{1/q} \\ &\leq \left(\sup_j \|D_j\| \right) \|(x_j)\|_{l_p(X)}, \end{aligned}$$

since $q \geq p$. Clearly $\|D\| \geq \sup \|D_j\|$.

Let E_j be the natural inclusion map from X into the elements of $l_p(X)$ which are zero except in the j th coordinate and P_j be the projection from $l_q(Y)$ onto Y given by choosing the j th coordinate. For all j , let $x_j \in B_X$, such that

$$\|P_j T E_j x_j\| = \|D_j x_j\| \geq \|D_j\|/2.$$

Because we began with a compact operator, $\{T E_j x_j\}$ is relatively compact in $l_q(Y)$. If $1 \leq q < \infty$, $\|P_j T E_j x_j\|$ converges to 0 because $T(B_{l_p(X)})$ is relatively compact and thus $\sum \|P_j y\|_Y^q$ converges uniformly for $y \in T(B_{l_p(X)})$. If $q = \infty$, then $p > 1$ and $(E_j x_j)$ converges weakly to 0. Because T is compact, $(T E_j x_j)$ converges in norm to 0. Therefore the limit of the norms of the operators D_j must be 0. Conversely, any sequence of operators (T_i) with $T_i \in \mathcal{K}(X, Y)$ for all i and $\lim \|T_i\| = 0$, induces a diagonal operator in $\mathcal{K}(l_p(X), l_q(Y))$, by $D(x_j) = (T_j x_j)$. Clearly each truncation $D^{(n)}$ of D ,

$$D^{(n)}(x_j) = (T_1 x_1, \dots, T_n x_n, 0, 0, \dots),$$

is compact and $D^{(n)}$ converges to D in norm. □

Remark 5.2. The conditions on p and q are not necessary for the proof that the space of diagonals of the compact operators is the range of a contractive map. In fact ℓ_p and ℓ_q can be replaced by spaces with unconditional basis. The computation of the bound on the norm of the diagonal operator requires that the norm on the domain dominate the norm on the range. In addition to prove compactness of the diagonal we used the fact that the norm of the tail of an element in $l_q(Y) \cap T(B_{l_p(X)})$ goes to zero. If $p = 1$ and $q = \infty$, this may fail and the diagonal of a compact operator may not be compact. An example of this is the one dimensional operator $T : \ell_1 \rightarrow \ell_\infty$ defined by $T(a_j) = (\sum a_j) 1_{\mathbb{N}}$. The corresponding diagonal operator is the inclusion map, $J(a_j) = \sum a_j 1_{\{j\}}$.

Theorem 5.3. *Let X and Y be Banach spaces and $1 \leq p < \infty$ and $p \leq q \leq \infty$ with $(p, q) \neq (1, \infty)$. Then we have*

$$\mathcal{K}(l_p(X), l_q(Y)) \sim C(\omega, \mathcal{K}(l_p(X), l_q(Y))).$$

Proof. By Lemma 5.1 with $X_1 = l_p(X)$ and $Y_1 = l_q(Y)$ in place of X and Y , we see that

$$C_0(\omega, \mathcal{K}(l_p(X), l_q(Y))) = C_0(\omega, \mathcal{K}(X_1, Y_1)) \xrightarrow{c} \mathcal{K}(X_1, Y_1) = \mathcal{K}(l_p(X), l_q(Y)).$$

Therefore by Pełczyński decomposition method [20, page 54] and 2.1 we infer

$$\mathcal{K}(l_p(X), l_q(Y)) \sim C_0(\omega, \mathcal{K}(l_p(X), l_q(Y))) \sim C(\omega, \mathcal{K}(l_p(X), l_q(Y))).$$

□

Corollary 5.4. $C(\beta\mathbb{N}, \ell_q)$ is isomorphic to $C(\omega \times \beta\mathbb{N}, \ell_q)$, for $1 \leq q < \infty$.

Proof. Let $p = 1$, $X = \ell_1$ and $Y = \ell_q$ in the previous result. Recall that $\mathcal{K}(\ell_1, \ell_q)$ is isomorphic to $C(\beta\mathbb{N}, \ell_q)$. Thus

$$C(\beta\mathbb{N}, \ell_q) \sim C(\omega, C(\beta\mathbb{N}, \ell_q)) \sim C(\omega \times \beta\mathbb{N}, \ell_q).$$

□

An analysis of the proof for the special case $p = 1$ shows that we can prove a version of the Cembranos-Freniche result for the case $K = \beta\mathbb{N}$.

Proposition 5.5. Suppose that X is a Banach space, $K < \infty$, and (P_j) is a sequence of projections defined on X with range X_j and $\|P_j\| \leq K$ for all j such that $\lim_j \|P_j x\| = 0$ for all $x \in X$. Then $(\sum_j C(\beta\mathbb{N}, X_j))_{c_0}$ is complemented in $C(\beta\mathbb{N}, X)$.

Proof. Let $\{N_j\}$ be a partition of \mathbb{N} into countably many disjoint infinite sets. Define an operator P on $C(\beta\mathbb{N}, X)$ by $(Pf)(n) = P_j(f(n))$ for all $f \in C(\beta\mathbb{N}, X)$, for all $n \in N_j$, $j = 1, 2, \dots$. We will show that Pf has norm relatively compact range and thus extends to a continuous function on $\beta\mathbb{N}$. The bound K on the norms of the projections shows that the range of Pf is bounded in X . To see that the range is totally bounded, let $\epsilon > 0$ and $\{x_m : m \in M\}$ be a finite $(\epsilon/4)K^{-1}$ -net in $f(\beta\mathbb{N})$. For each $m \in M$, there is an integer J_m such that for all $j \geq J_m$,

$$\|P_j x_m\| < \epsilon/4.$$

Let $J = \max_m J_m$. If $j \geq J$ and $x \in f(\beta\mathbb{N})$ then

$$\|P_j x\| \leq \min_m (\|P_j x_m\| + \|P_j(x_m - x)\|) < \epsilon/2.$$

Thus the range of Pf is contained in

$$\cup_{j \leq J} P_j(f(\beta\mathbb{N})) \cup \frac{\epsilon}{2} B_X,$$

and an $\epsilon/2$ -net in the compact set $\cup_{j \leq J} P_j(f(\beta\mathbb{N}))$ will yield an ϵ -net.

Clearly P will be linear, bounded and the identity on $C(\beta N_j, X_j)$ for all j . Moreover the argument above shows that

$$Pf \in \left(\sum_j C(\beta N_j, X_j) \right)_{c_0}.$$

□

Remark 5.6. The results in these sections allow us to point out some limitations of our approach if one considers more general classification problems for the spaces $C(K, X)$. Notice that $C(\beta\mathbb{N}, \ell_2 \oplus c_0)$ is isomorphic to

$$C(\beta\mathbb{N}, \ell_2) \oplus C(\beta\mathbb{N}, c_0) \sim C(\omega \times \beta\mathbb{N}, \ell_2) \oplus C(\omega \times \beta\mathbb{N}) \sim C(\beta\mathbb{N}, \ell_2).$$

If we instead use $C(\omega^\omega)$, we get a different outcome.

$$C(\beta\mathbb{N}, \ell_2 \oplus C(\omega^\omega)) \sim C(\beta\mathbb{N}, \ell_2) \oplus C(\beta\mathbb{N}, C(\omega^\omega)).$$

This space is not isomorphic to $C(\beta\mathbb{N}, \ell_2)$. It is complemented in $C(\beta\mathbb{N} \times \omega^\omega, \ell_2)$ but it does not seem likely that it contains $C(\omega^\omega, \ell_2)$ as a complemented subspace. It also seems doubtful that there is any compact Hausdorff space K such that $C(\beta\mathbb{N}, \ell_2 \oplus C(\omega^\omega))$ is isomorphic to $C(K, \ell_2)$.

6. OPEN QUESTIONS

We end this paper by stating some questions which it raises. We do not know whether the statement of our main result (Theorem 3.7) remains true in the case where $X = l_\infty$, that is,

Problem 6.1. *Let K_1 and K_2 be infinite compact metric spaces. Does it follow that*

$$C(\beta\mathbb{N} \times \beta\mathbb{N} \times K_1) \sim C(\beta\mathbb{N} \times \beta\mathbb{N} \times K_2) \implies C(K_1) \sim C(K_2)?$$

Notice that with $X = \mathbb{R}$ in Theorem 3.7 we have (see also [10, Theorem 5.7])

$$C(\beta\mathbb{N} \times K_1) \sim C(\beta\mathbb{N} \times K_2) \implies C(K_1) \sim C(K_2),$$

for any infinite compact metric spaces K_1 and K_2 .

The case $K_1 = \omega$ and $K_2 = \{1\}$ of Problem 6.1 is the statement of Theorem 5.4 when $q = \infty$, that is,

Problem 6.2. *Is it true that*

$$C(\omega \times \beta\mathbb{N} \times \beta\mathbb{N}) \sim C(\beta\mathbb{N} \times \beta\mathbb{N})?$$

This is a special case of the following question for which Theorem 5.4 gives some answers:

Problem 6.3. *For which infinite dimensional Banach spaces X is*

$$C(\omega \times \beta\mathbb{N}, X) \sim C(\beta\mathbb{N}, X)?$$

Of course if X is a finite dimensional space, this is false. If X is isomorphic to $C(\omega, Y)$, then

$$C(\omega \times \beta\mathbb{N}, X) \sim C(\beta\mathbb{N}, C(\omega \times \omega, Y)) \sim C(\beta\mathbb{N}, C(\omega, Y)),$$

and thus such X are examples.

One way to approach Problem 6.2 is to study the isomorphic classification of the complemented subspaces of $C(\beta\mathbb{N} \times \beta\mathbb{N})$. Thus, it would be interesting to solve the following intriguing problem which is a particular case of the well known complemented subspace problem for $C(K)$ spaces, see for instance [24, section 5]

Problem 6.4. *Let X be a complemented subspace of $C(\beta\mathbb{N} \times \beta\mathbb{N})$. Suppose that X is infinite dimensional separable space. Is X isomorphic to c_0 ?*

In particular, the other separable $C(K)$ spaces would be eliminated if the answer to the following is no.

Problem 6.5. *Is it true that $C(\omega^\omega) \xhookrightarrow{c} C(\beta\mathbb{N} \times \beta\mathbb{N})$?*

Finally, observe that Theorem 3.2 leads naturally to the following problem which is in connection with the Cembranos and Freniche's theorem (Theorem 1.2)

Problem 6.6. *Suppose that X is a Banach space and K is an infinite compact space. Is it true that*

- (1) $C(\omega^\omega) \xhookrightarrow{c} C(K, X) \implies c_0 \xhookrightarrow{c} C(K)$ or $c_0 \hookrightarrow X$?
- (2) $C(\omega^\omega) \xhookrightarrow{c} C(\beta\mathbb{N}, X) \implies c_0 \xhookrightarrow{c} X$?

If X is separable, then c_0 is always complemented so the latter is true by Theorem 3.2. As noted after the proof of that theorem variations of this problem with either $C(\omega^\omega)$ or uniformly complemented copies of (ω^n) could also be considered.

It is possible that the proper context for this line of investigation is actually injective tensor products.

Problem 6.7. *Suppose that X and Y are Banach spaces, $\alpha > 0$ and $\alpha_n \uparrow \omega^\alpha$ are ordinals, and $C(\omega^{\omega^\alpha})$ isomorphic to a (complemented) subspace of $X \overset{\vee}{\otimes} Y$, is*

- $C(\omega^{\omega^\alpha})$ isomorphic to a (complemented) subspace of X or Y
or
- $C(\omega^{\alpha_n})$ uniformly isomorphic to (complemented) subspace of one of X and Y and c_0 isomorphic to a subspace of the other?

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER OK 74078, USA

E-mail address: `alspach@math.okstate.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SÃO PAULO, SÃO PAULO, BRAZIL 05508-090

E-mail address: `eloi@ime.usp.br`